# Pearls of Algorithms

## Part 2: Randomized Algorithms and Probabilistic Analysis

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# Efficient Algorithms

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**Engineer**

The algorithm must be efficient in practice, i.e., it must solve practical instances in an appropriate amount of time.

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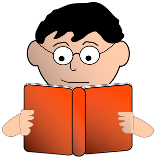
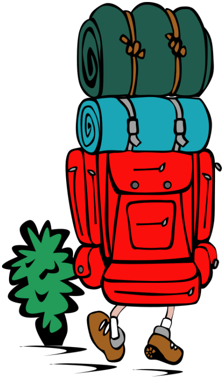


**Theorist**
The algorithm must be efficient in the worst case, i.e., it must solve all instances in polynomial time.

# The Knapsack Problem



## Knapsack problem (KP)

- **Input**
  - set of items $\{1, \ldots, n\}$
  - profits $p_1, \ldots, p_n$
  - weights $w_1, \ldots, w_n$
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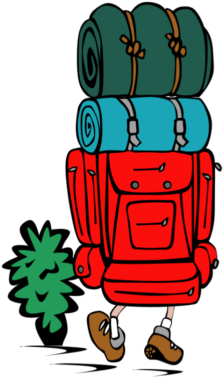

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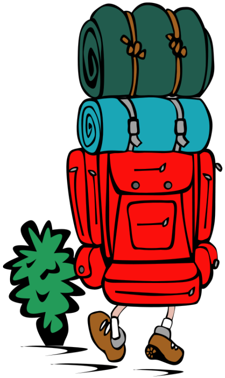

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  Find a subset of items that fits into the knapsack and maximizes the profit.
- **Formal description**

$$\begin{aligned} \text{max} \quad & p_1 x_1 + \dots + p_n x_n \\ \text{subject to} \quad & w_1 x_1 + \dots + w_n x_n \le b \\ & \text{and } x_i \in \{0, 1\} \end{aligned}$$

# Different Opinions

**Theorists say...**

- KP is NP-hard.
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**Engineers say...**

- KP is easy to solve!
- Does not even require quadratic time.

There are very good heuristics for practical instances of KP.

# Reason for discrepancy

## Reason for discrepancy

- Worst-case complexity is too pessimistic!
- There are (artificial) worst-case instances for KP on which the heuristics are not efficient. These instances, however, do not occur in practice.
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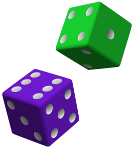
## Average-case analysis

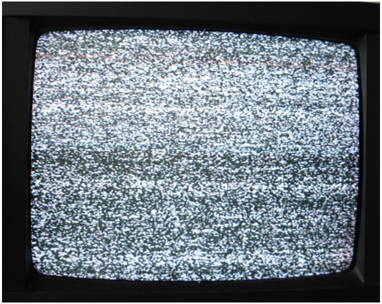


Is it realistic to consider the average case behavior instead of the worst case behavior?

# Random Inputs are not Typical

**Random inputs are not typical!**

If real-world data was random, watching TV would be very boring...



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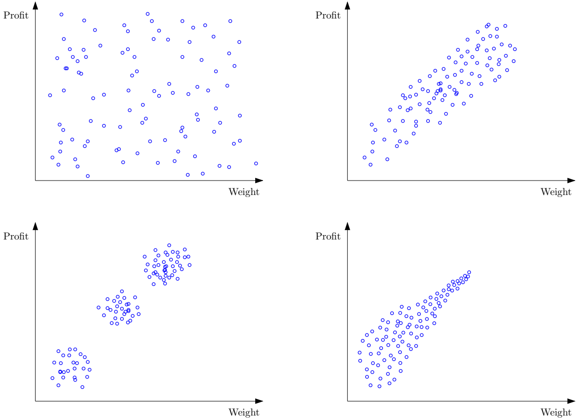
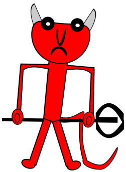
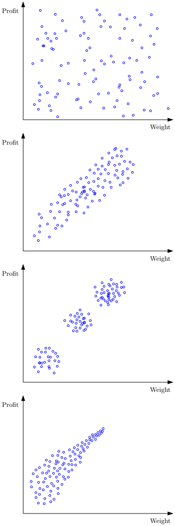



It depends very much on the concrete application. We cannot say in general what a typical instance for KP looks like.

# Smoothed Analysis

Step 1:
Adversary
chooses input $I$.





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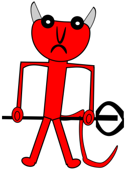

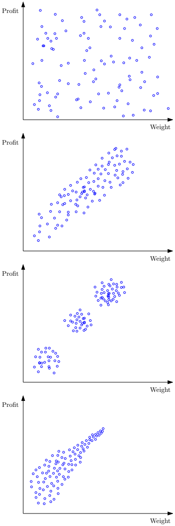


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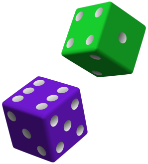
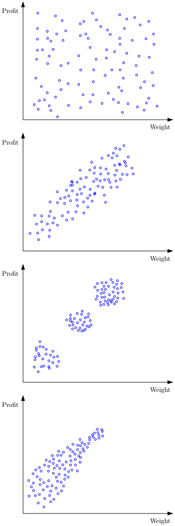
perturbation.
$I \to \mathrm{per}(I)$





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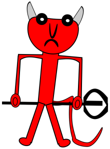

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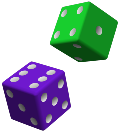

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Smoothed Complexity = worst expected running time the adversary can achieve

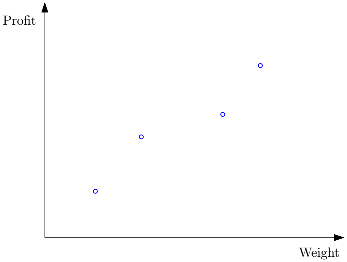
**Why do we consider this model?**

- First step alone: worst case analysis.
- Second step models random influences, e.g., measurement errors, numerical imprecision, rounding, ...
- So we have a combination of both: instances of any structure with some amount of random noise.

# Perturbation

## Example

- Step 1: Adversary chooses all $p_i, w_i \in [0, 1]$ arbitrarily.



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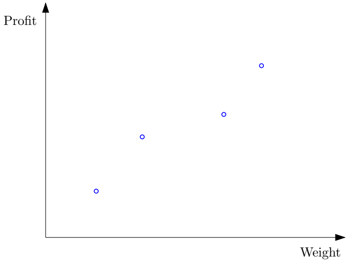


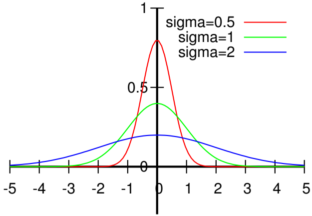

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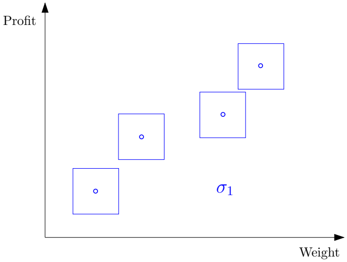


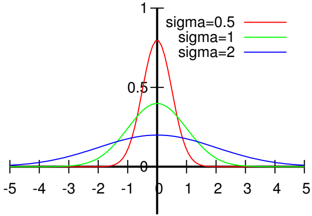

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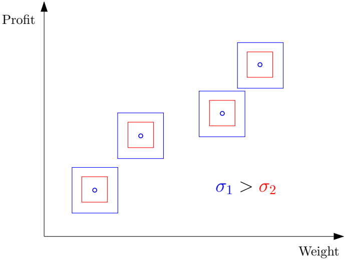
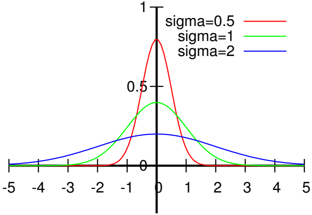
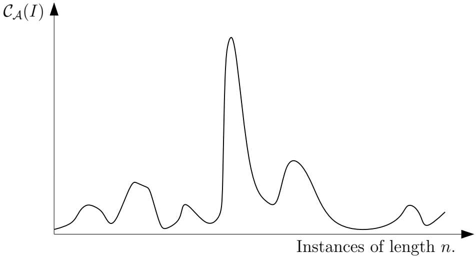


# Complexity Measures

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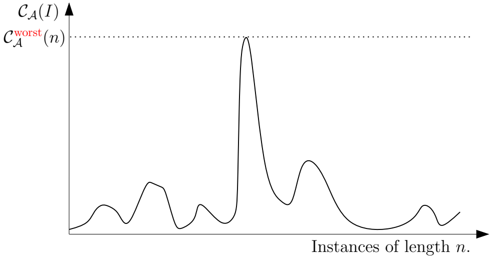


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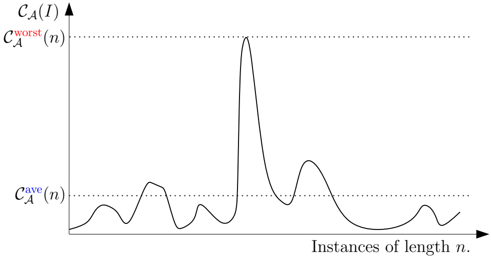


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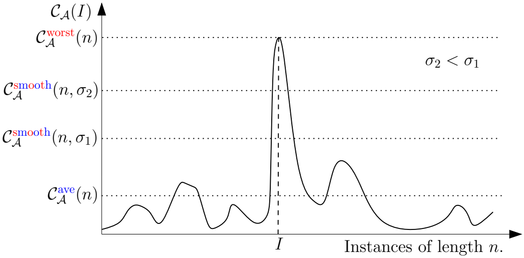


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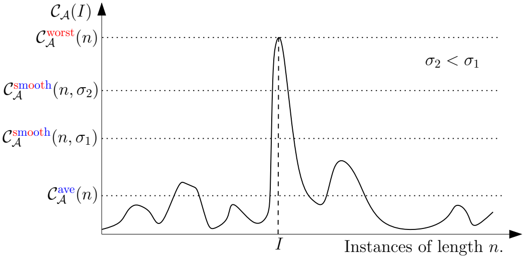


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smoothed complexity low $\Rightarrow$ bad instances are isolated peaks

# Linear Programs

## Linear Programs (LPs)

- variables: $x_1, \dots, x_d \in \mathbb{R}$.
- linear objective function:
  $\max c_1 x_1 + \ldots + c_n x_n$.

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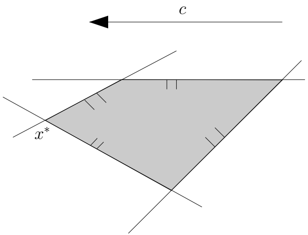

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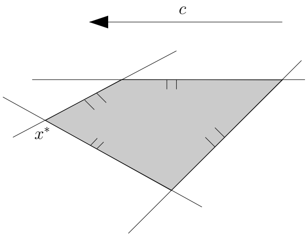
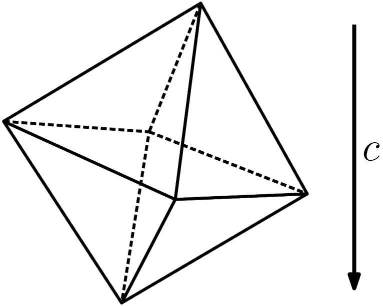



## Complexity of LPs

LPs can be solved in polynomial time by the ellipsoid method [Khachiyan 1979].

# Simplex Algorithm




**Simplex Algorithm**

- The simplex method walks along the vertices of the polytope in the direction of the objective function $c^T x$.

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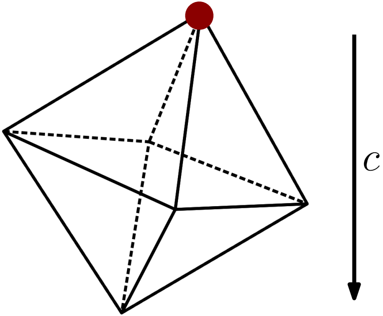


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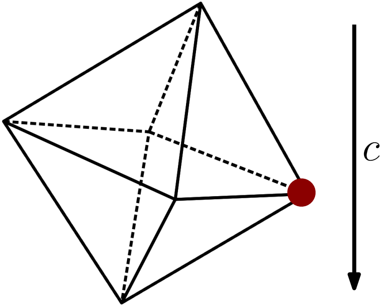


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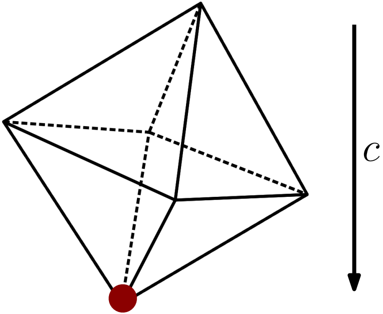


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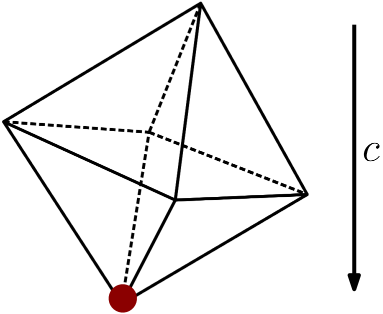


**Simplex Algorithm**

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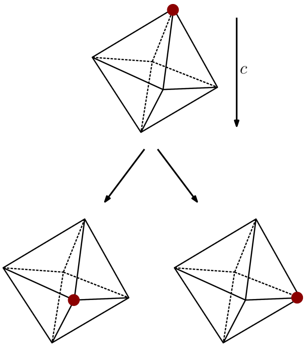
- Exponential in the worst case.
- Works well in practice.

# Pivot Rules



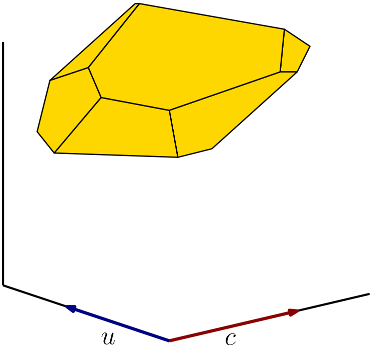

## Pivot Rules

- How is a better vertex on the polytope chosen if there are multiple options?
- Different pivot rules have been suggested:
  - random
  - steepest descent
  - shadow vertex pivot rule
  - ...

# Shadow Vertex Pivot Rule

## Shadow Vertex Pivot Rule

- Let $x_0$ be some vertex of the polytope.
- Compute $u \in \mathbb{R}^d$ such that $x_0$ maximizes $u^T x$.



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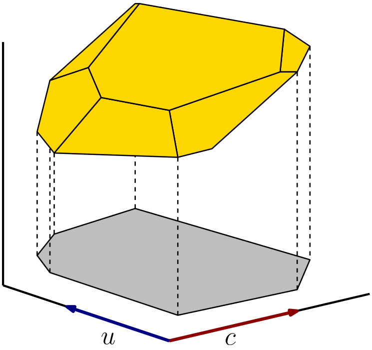
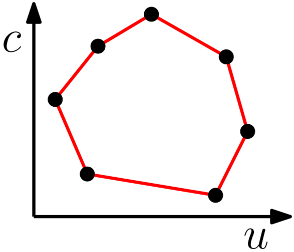
- Let $x_0$ be some vertex of the polytope.
- Compute $u \in \mathbb{R}^d$ such that $x_0$ maximizes $u^T x$.
- Project the polytope onto the plane spanned by $c$ and $u$.



# Shadow Vertex Pivot Rule

## 2-dimensional projection

- The projection is 2-dimensional, that is, a polygon.



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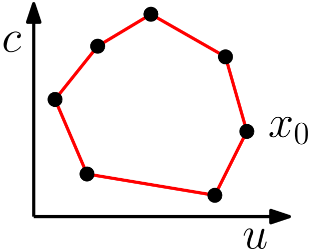

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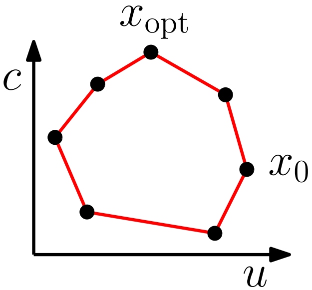

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- $x_0$ is a vertex of the polygon.
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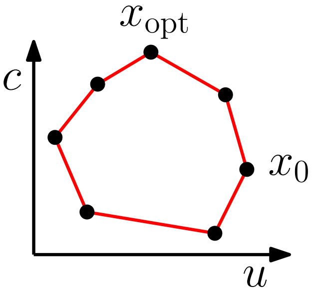
- Edges of the polygon correspond to edges of the polytope.



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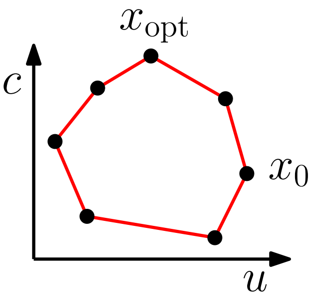

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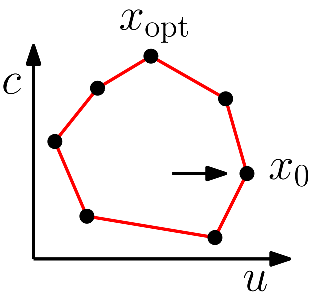

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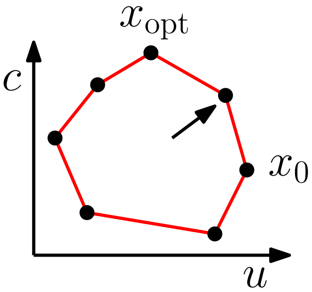

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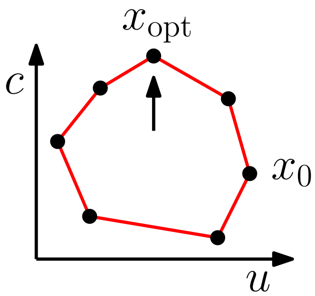
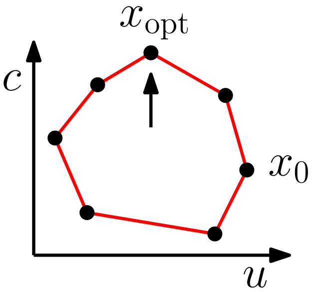


# Shadow Vertex Pivot Rule

## 2-dimensional projection

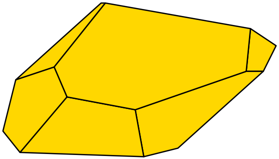
- In 2 dimension the simplex method is easy; it just follows the edges of the polygon.
- It starts at $x_0$...
- ... and follows the edges to $x_{\text{opt}}$.
- The polygon can have an exponential number of edges.



# Perturbed Linear Programs

## Perturbed LPs

- Step 1: Adversary specifies arbitrary LP:
  $\max c^T x$ subject to $a_1^T x \leq b_1 \ldots a_n^T x \leq b_n$.
  W. l. o. g. $\|(a_i, b_i)\| = 1$.



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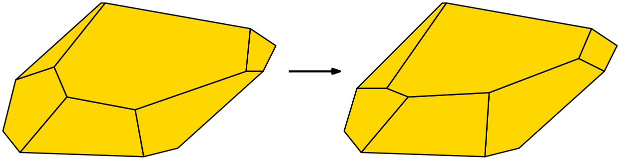
- Step 2: Add Gaussian random variable with standard deviation $\sigma$ to each coefficient in the constraints.



# Smoothed Analysis of the Simplex Algorithm

**Theorem [Spielman and Teng 2001]**

The expected number of edges on the polygon is

$$O\left(\mathrm{poly}\left(n, d, \sigma^{-1}\right)\right) \quad .$$

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Running time is polynomial in $n$, $d$, and $\sigma^{-1}$.

Already for small perturbation it is extremely unlikely to hit a bad instance.

# Improved Analysis

**Theorem [Vershynin 2006]**

The smoothed running time of the simplex algorithm with shadow vertex pivot rule is

$$O\left(\mathrm{poly}\left(\log n, d, \sigma^{-1}\right)\right) \quad .$$

Running time is only polylogarithmic in the number of constraints $n$.

# Overview of the coming Lectures

**Smoothed Analysis**

- 2-Opt heuristic for the traveling salesperson problem
- Nemhauser/Ullmann algorithm for the knapsack problem
- $k$-means clustering